

Optimal robust smoothing extragradient algorithms for stochastic variational inequality problems

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Abstract—We consider stochastic variational inequality problems where the mapping is monotone over a compact convex set. We present two robust variants of stochastic extragradient algorithms for solving such problems. Of these, the first scheme employs an iterative averaging technique where we consider a generalized choice for the weights in the averaged sequence. Our first contribution is to show that using an appropriate choice for these weights, a suitably defined gap function attains the optimal rate of convergence $\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$. In the second part of the paper, under an additional assumption of weak-sharpness, we update the stepsize sequence using a recursive rule that leverages problem parameters. The second contribution lies in showing that employing such a sequence, the extragradient algorithm possesses almost-sure convergence to the solution as well as convergence in a mean-squared sense to the solution of the problem at the rate $\mathcal{O}\left(\frac{1}{k}\right)$. Motivated by the absence of a Lipschitzian parameter, in both schemes we utilize a locally randomized smoothing scheme. Importantly, by approximating a smooth mapping, this scheme enables us to estimate the Lipschitzian parameter. The smoothing parameter is updated per iteration and we show convergence to the solution of the original problem in both algorithms.

I. INTRODUCTION

The theory of variational inequality (VI) was introduced in mid-1960s, motivated by the elastostatic equilibrium problems. During the past five decades, this subject has been a powerful framework in modeling a wide range of optimization and equilibrium problems in operations research, engineering, finance, and economics (cf. [1], [2]). Given a set $X \subset \mathbb{R}^n$ and a mapping $F : X \rightarrow \mathbb{R}^n$, a VI problem, denoted by $\text{VI}(X, F)$, requires determining an $x^* \in X$ such that $F(x^*)^T(x - x^*) \geq 0$ for any $x \in X$. In this paper, our interest lies in computation of solutions to VI problems with uncertain settings. We consider the case where $F : X \rightarrow \mathbb{R}^n$ represents the expected value of a stochastic mapping $\Phi : X \times \Omega \rightarrow \mathbb{R}^n$, i.e., $F_i(x) \triangleq \mathbb{E}[\Phi_i(x, \xi(\omega))]$ where $\xi : \Omega \rightarrow \mathbb{R}^d$ is a d -dimensional random variable and $(\Omega, \mathcal{F}, \mathbb{P})$ denotes the associated probability space. Consequently, $x^* \in X$ solves $\text{VI}(X, F)$ if

$$\mathbb{E}[\Phi(x^*, \xi(\omega))]^T(x - x^*) \geq 0, \text{ for every } x \in X. \quad (1)$$

The stochastic VI problem (1) arises in many situations, often modeling stochastic convex optimization problems and stochastic Nash equilibrium problems. Utilized by sampling

from an unbiased stochastic oracle $\Phi(x, \xi)$, iterative algorithms have been developed to solve the problem (1). Such schemes include the stochastic approximation type methods [3], [4] and extragradient type methods [5], [6], [7]. In a recent work [8], Kannan and Shanbhag studied almost-sure convergence of extragradient algorithms and provided sufficiency conditions for the solvability of stochastic VIs with pseudo-monotone mappings. Prox-type methods were first developed by Nemirovski [7] for solving VIs with monotone and Lipschitz mappings and addressed different problems in convex optimization and variational inequalities. Recently, Juditsky et al. [5] introduced the stochastic Mirror-Prox (SMP) algorithm for solving stochastic VIs in both smooth and nonsmooth cases. In [5], it is assumed that the mapping F is monotone and satisfies the following relation:

$$\|F(x) - F(y)\| \leq L\|x - y\| + B, \quad \text{for all } x, y \in X,$$

where $L \geq 0$ and $B \geq 0$ are known constants. Under such conditions, by choosing a constant stepsize rule $0 < \gamma < \frac{1}{\sqrt{3L}}$, the optimal rate of convergence of a suitably defined gap function is shown to be $\mathcal{O}(1) \left(\frac{L}{t} + \frac{B+\sigma}{\sqrt{t}} \right)$ where σ is the upper bound on the variance of the stochastic oracle and t is the pre-fixed number of iterations. In this paper, our main goal is developing two classes of robust extragradient algorithms for monotone stochastic VIs in extension of the work in [5] and [8]. The first class of the proposed extragradient algorithms employs the well-known averaging technique utilized by new choices of the averaging weights. In the second part of the paper, we consider monotone VI problems with weak-sharpness property and we develop an extragradient algorithm that employs a recursive stepsize policy. Such a stepsize sequence is obtained in terms of problem parameters. The word “robust” refers to the self-tuned stepsize rule and capability of dealing with the absence of a Lipschitz constant. Our main contributions in this paper are described as follows:

(1) *Choice of the averaging weights*: The SMP algorithm in [5] generates a weighted-iterative averaging sequence of the form $\bar{x}_t \triangleq \sum_{i=1}^k \frac{\gamma_i}{\sum_{i=1}^k \gamma_i} x_t$ where x_t is generated at the t -th iteration and $\gamma_t > 0$ is the corresponding stepsize. Recently, Nedić and Lee [9] showed that using different weights of the form $\frac{\gamma_t^{-1}}{\sum_{i=1}^k \gamma_i^{-1}}$, the subgradient mirror-descent algorithms attain the optimal rate of convergence without requiring window-based averaging sequences similar to [10] and [3]. In this paper, we generalize this idea in two directions: First, we show that such choices can be applied in the stochastic extragradient algorithms (e.g. [5]). Second, using the weights $\frac{\gamma_t^r}{\sum_{i=1}^k \gamma_i^r}$ where $r \in \mathbb{R}$ is a constant, we show that for any $r < 1$, the optimal convergence rate is attained. Note that this optimal rate cannot be attained when $r = 1$ (e.g. in [5]).

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(2) *Developing parameterized stepsizes:* In the second part of the paper, we assume that the problem with monotone mapping has a weak-sharpness property. We develop a recursive stepsize sequence that leverages problem parameters and show that by employing such a stepsize rule, the sequence $\{x_t\}$ generated by the extragradient algorithm converges in almost-sure sense to the solution of the problem. Moreover, we show that this robust scheme converges in mean-squared sense to the solution of the problem with the rate of $\mathcal{O}(\frac{1}{k})$.

(3) *Estimating the Lipschitzian parameter L :* While the SMP algorithm in [5] addresses both smooth and nonsmooth problems by allowing L to be zero, knowing the constant L benefits the rate of convergence since the term $\frac{L}{t}$ decays faster than $\frac{B+\sigma}{\sqrt{t}}$. Moreover, the prescribed constant stepsize in [5] is bounded in terms of L . We consider the case that the mapping is either nonsmooth or the constant L is unknown. Motivated by a smoothing technique first introduced by Steklov [11], our goal is addressing these cases. Recently in [12] and [4], by employing this technique we addressed nonsmoothness in developing adaptive stepsizes stochastic approximation schemes in absence or unavailability of a Lipschitz constant. We extend those results by applying such smoothing technique for both extragradient schemes.

The paper is organized as follows: in Section II, we present an algorithm utilizing the weighted averaging and we show the convergence of a suitably defined gap function to zero with the optimal rate. In Section III we present an algorithm with a recursive stepsize updates and provide its convergence and rate analysis. We conclude with some remarks in Section IV.

Notation: In this paper, a vector x is assumed to be a column vector, x^T denotes the transpose of a vector x , and $\|x\|$ denotes the Euclidean vector norm, i.e., $\|x\| = \sqrt{x^T x}$. We let $\Pi_X(x)$ denote the Euclidean projection of a vector x on a set X , i.e., $\|x - \Pi_X(x)\| = \min_{y \in X} \|x - y\|$. We write a.s. as the abbreviation for “almost surely”. We use $\mathbb{E}[z]$ to denote the expectation of a random variable z .

II. STOCHASTIC EXTRAGRADIENT METHOD

We consider the following stochastic variant of Korpelevich’s extragradient scheme: for all $t \geq 0$:

$$\begin{aligned} y_{t+1} &= \Pi_X [x_t - \gamma_t \Phi(x_t + z'_t, \xi'_t)], \\ x_{t+1} &= \Pi_X [x_t - \gamma_t \Phi(y_{t+1} + z_t, \xi_t)]. \end{aligned} \quad (2)$$

In this scheme, $\{\gamma_t\}$ is the stepsize sequence and $x_0 \in X$ is a random initial point with $\mathbb{E}[\|x_0\|^2] \leq \infty$. The vectors ξ_t and ξ'_t are two i.i.d samples from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also, z_t and z'_t are two i.i.d. samples from a uniform random variable $Z_t \in \mathbb{R}^n$. The i th element of Z_t , denoted by $Z_{t,i}$ is uniformly distributed in the interval $[-\frac{\epsilon_t}{2}, \frac{\epsilon_t}{2}]$. To have a well-defined algorithm, we define the set $X^\epsilon \triangleq X + C_n(0, \epsilon)$, where $C_n(0, \epsilon) \triangleq \{x \in \mathbb{R}^n \mid x_i \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}]\}$ is a cube centered at origin. The scalar ϵ is assumed to be an upper bound for the sequence $\{\epsilon_t\}$. We denote the history of the scheme using the following notations:

$$\begin{aligned} \mathcal{F}'_t &\triangleq \{x_0\} \cup \{\xi'_0, z'_0, \xi'_1, z'_1, \dots, \xi'_t, z'_t\}, \\ \mathcal{F}_t &\triangleq \{\xi_0, z_0, \xi_1, z_1, \dots, \xi_t, z_t\}, \end{aligned} \quad (3)$$

where $t \geq 0$. The first set of assumptions is on the set X , the mapping F , and the random variables.

Assumption 1: Let the following hold:

(a) The set $X \subset \mathbb{R}^n$ is closed, convex, and bounded, i.e., $\|x\| \leq M$ for all $x \in X$ and some $M > 0$. (b) The mapping F is monotone and bounded on the set X^ϵ , i.e., $\|F(x)\| \leq C$ for all $x \in X^\epsilon$ and some $C > 0$. (c) There exists an $x^* \in X$ such that $(x - x^*)^T \mathbb{E}[\Phi(x^*, \xi)] \geq 0$, for all $x \in X$. (d) The random variables z_t, z'_t, ξ_k and ξ'_k are all i.i.d. and independent from each other for any $t, k \geq 0$.

We also make use of the following assumptions.

Assumption 2: Define $w(x) \triangleq \Phi(x, \xi) - F(x)$ for $x \in X^\epsilon$. We assume, the samples $\Phi(x, \xi)$ taken in algorithm (2) are unbiased, i.e., $\mathbb{E}[w(x)] = 0$ for all $x \in X^\epsilon$. Moreover, the variance of the samples $\Phi(x, \xi)$ is bounded, i.e., there is $\nu > 0$ such that $\mathbb{E}[\|w(x)\|^2] \leq \nu^2$ for all $x \in X^\epsilon$.

Also, we define the approximate mapping

$$F_t(x) \triangleq \mathbb{E}[F(x + Z_t)], \quad \text{for all } x \in X \text{ and all } t \geq 0. \quad (4)$$

The following result has been shown in our prior work [13] (on random local smoothing):

Lemma 1: Let the mapping $F_t : X \rightarrow \mathbb{R}^n$ be defined by (4) where Z_t is uniformly distributed over $C_n(0, \epsilon_t)$. Then, for all $t \geq 0$, F_t is Lipschitz continuous over the set X , i.e.,

$$\|F_t(x) - F_t(y)\| \leq \frac{\sqrt{n}C}{\epsilon_t} \|x - y\|, \quad \text{for all } x, y \in X.$$

In our analysis, we exploit the following properties of the projection mapping (cf. Chapter 2 in [14]).

Lemma 2 (Properties of the projection mapping): Let $X \subseteq \mathbb{R}^n$ be a nonempty closed convex set.

- (a) $\|\Pi_X[u] - \Pi_X[v]\| \leq \|u - v\|$, for all $u, v \in \mathbb{R}^n$.
- (b) $(\Pi_X[u] - u)^T (x - \Pi_X[u]) \geq 0$, for all $u \in \mathbb{R}^n$ and $x \in X$.

For notational convenience, we define the stochastic errors of algorithm (2) as follows:

$$\begin{aligned} w_t &\triangleq \Phi(y_{t+1} + z_t, \xi_t) - F(y_{t+1} + z_t), \\ w'_t &\triangleq \Phi(x_t + z'_t, \xi'_t) - F(x_t + z'_t), \\ \Delta_t &\triangleq F(y_{t+1} + z_t) - F_t(y_{t+1}), \\ \Delta'_t &\triangleq F(x_t + z'_t) - F_t(x_t). \end{aligned} \quad (5)$$

We have the following basic result for the algorithm.

Lemma 3: Let Assumptions 1 and 2 hold, and $0 < \gamma_t \leq \frac{\epsilon_t}{\sqrt{5n}C}$ for all $t \geq 0$. Then, for the iterates of algorithm (2), the following relation holds for all $y \in X$ and all $t \geq 0$:

$$\begin{aligned} &\|x_{t+1} - y\|^2 \\ &\leq \|x_t - y\|^2 + 2\gamma_t F(y_{t+1} + z_t)^T (y - (y_{t+1} + z_t)) \\ &\quad + 2\sqrt{n}C\gamma_t \epsilon_t + 2\gamma_t w_t^T (y - y_{t+1}) + 5\gamma_t^2 B_t, \end{aligned} \quad (6)$$

where $B_t \triangleq \|\Delta_t\|^2 + \|\Delta'_t\|^2 + \|w_t\|^2 + \|w'_t\|^2$, and Δ_t, Δ'_t, w_t and w'_t are as defined in (5).

Proof: Let $y \in X$ and $t \geq 0$ be fixed, but arbitrary. We start by estimating an upper bound for the stochastic term

$\|x_{t+1} - y\|^2$. From algorithm (2), we have the following:

$$\begin{aligned} \|x_{t+1} - y\|^2 &= \|x_{t+1} - x_t + x_t - y\|^2 \\ &= \|x_{t+1} - x_t\|^2 + \|x_t - y\|^2 + 2(x_{t+1} - x_t)^T(x_t - y) \\ &= \|x_{t+1} - x_t\|^2 + \|x_t - y\|^2 \\ &\quad + 2(x_{t+1} - x_t)^T(x_t - x_{t+1}) + 2(x_{t+1} - x_t)^T(x_{t+1} - y) \\ &= \|x_t - y\|^2 - \|x_{t+1} - x_t\|^2 + 2(x_{t+1} - x_t)^T(x_{t+1} - y) \end{aligned} \quad (7)$$

where in the second equality, we add and subtract x_t , and in the third equality, we add and subtract x_{t+1} . Consider Lemma 2(b), and let $u \triangleq x_t - \gamma_t \Phi(y_{t+1} + z_t, \xi_t)$ and $x \triangleq y$. By algorithm (2), we have $x_{t+1} = \Pi_X[u]$. Thus, by Lemma 2(b), we obtain

$$\begin{aligned} 0 &\leq (x_{t+1} - (x_t - \gamma_t \Phi(y_{t+1} + z_t, \xi_t)))^T(y - x_{t+1}) \\ &= (x_{t+1} - x_t)^T(y - x_{t+1}) + \gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1}). \end{aligned}$$

Hence $(x_{t+1} - x_t)^T(x_{t+1} - y) \leq \gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1})$, and by relation (7), it follows that

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|x_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1}). \end{aligned}$$

By adding and subtracting y_{t+1} in $\|x_{t+1} - x_t\|^2$, we have

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|x_{t+1} - y_{t+1} + y_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1}) \\ &= \|x_t - y\|^2 - \|x_{t+1} - y_{t+1}\|^2 \\ &\quad - \|y_{t+1} - x_t\|^2 - 2(x_{t+1} - y_{t+1})^T(y_{t+1} - x_t) \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1}). \end{aligned} \quad (8)$$

Using Lemma 2(b) with $u \triangleq x_t - \gamma_t \Phi(x_t + z'_t, \xi'_t)$ and $x \triangleq x_{t+1}$, and using $y_{t+1} = \Pi_X[u]$ (see algorithm (2)), we obtain

$$\begin{aligned} 0 &\leq (y_{t+1} - (x_t - \gamma_t \Phi(x_t + z'_t, \xi'_t)))^T(x_{t+1} - y_{t+1}) \\ &= (y_{t+1} - x_t)^T(x_{t+1} - y_{t+1}) \\ &\quad + \gamma_t \Phi(x_t + z'_t, \xi'_t)^T(x_{t+1} - y_{t+1}). \end{aligned}$$

Therefore, $-(y_{t+1} - x_t)^T(x_{t+1} - y_{t+1}) \leq \gamma_t \Phi(x_t + z'_t, \xi'_t)^T(x_{t+1} - y_{t+1})$, which together with relation (8), yields

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|x_{t+1} - y_{t+1}\|^2 \\ &\quad - \|y_{t+1} - x_t\|^2 + 2\gamma_t \Phi(x_t + z'_t, \xi'_t)^T(x_{t+1} - y_{t+1}) \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - x_{t+1}). \end{aligned}$$

By adding and subtracting y_{t+1} in the appropriate terms of the preceding relation, we further have

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|x_{t+1} - y_{t+1}\|^2 \\ &\quad - \|y_{t+1} - x_t\|^2 + 2\gamma_t \Phi(x_t + z'_t, \xi'_t)^T(x_{t+1} - y_{t+1}) \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1} + y_{t+1} - x_{t+1}) \\ &= \|x_t - y\|^2 - \underbrace{\|x_{t+1} - y_{t+1}\|^2}_{\text{Term 1}} - \|y_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) \\ &\quad + \underbrace{2\gamma_t (\Phi(x_t + z'_t, \xi'_t) - \Phi(y_{t+1} + z_t, \xi_t))^T(x_{t+1} - y_{t+1})}_{\text{Term 2}}. \end{aligned} \quad (9)$$

From the definitions of Terms 1 and 2, by employing $2ab \leq a^2 + b^2$, for any $a, b \in \mathbb{R}$, we have that

$$\text{Term 1} + \text{Term 2} \leq \gamma_t^2 \|\Phi(y_{t+1} + z_t, \xi_t) - \Phi(x_t + z'_t, \xi'_t)\|^2.$$

The preceding inequality and relation (9) imply that

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|y_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) \\ &\quad + \gamma_t^2 \|\Phi(y_{t+1} + z_t, \xi_t) - \Phi(x_t + z'_t, \xi'_t)\|^2. \end{aligned} \quad (10)$$

We now estimate the term $\|\Phi(y_{t+1} + z_t, \xi_t) - \Phi(x_t + z'_t, \xi'_t)\|^2$. Using the definitions (5), we have

$$\begin{aligned} &\|\Phi(y_{t+1} + z_t, \xi_t) - \Phi(x_t + z'_t, \xi'_t)\|^2 \\ &= \|F(y_{t+1} + z_t) + w_t - F(x_t + z_t) - w'_t\|^2 \\ &= \|F_t(y_{t+1}) + \Delta_t + w_t - F_t(x_t) - \Delta'_t - w'_t\|^2. \end{aligned}$$

By the triangle inequality, we further obtain

$$\begin{aligned} &\|\Phi(y_{t+1} + z_t, \xi_t) - \Phi(x_t + z'_t, \xi'_t)\|^2 \\ &\leq (\|F_t(y_{t+1}) - F_t(x_t)\| + \|\Delta_t\| + \|\Delta'_t\| + \|w_t\| + \|w'_t\|)^2 \\ &\leq 5\|F_t(y_{t+1}) - F_t(x_t)\|^2 + 5B_t \\ &\leq 5\frac{nC^2}{\epsilon_t^2}\|y_{t+1} - x_t\|^2 + 5B_t, \end{aligned} \quad (11)$$

where second inequality is obtained from the following relation for any $a_1, a_2, \dots, a_m \in \mathbb{R}$ and any integer $m \geq 2$:

$$(a_1 + a_2 + \dots + a_m)^2 \leq m(a_1^2 + a_2^2 + \dots + a_m^2),$$

and the last inequality in (11) is obtained using the Lipschitzian property of mapping F_t from Lemma (1). Using the upper bound we found in (11), from inequality (10), we obtain

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 - \|y_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) \\ &\quad + 5nC^2 \frac{\gamma_t^2}{\epsilon_t^2} \|y_{t+1} - x_t\|^2 + 5\gamma_t^2 B_t \\ &\leq \|x_t - y\|^2 - \left(1 - 5nC^2 \frac{\gamma_t^2}{\epsilon_t^2}\right) \|y_{t+1} - x_t\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) + 5\gamma_t^2 B_t. \end{aligned}$$

From our assumption that $\gamma_t \leq \frac{\epsilon_t}{\sqrt{5nC}}$, we have $5nC^2 \frac{\gamma_t^2}{\epsilon_t^2} \leq 1$. Therefore, from the preceding inequality we obtain

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 \\ &\quad + 2\gamma_t \Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) + 5\gamma_t^2 B_t. \end{aligned} \quad (12)$$

From the definition of w_t in (5) we obtain

$$\begin{aligned} &\Phi(y_{t+1} + z_t, \xi_t)^T(y - y_{t+1}) \\ &= F(y_{t+1} + z_t)^T(y - y_{t+1}) + w_t^T(y - y_{t+1}) \\ &= F(y_{t+1} + z_t)^T(y - (y_{t+1} + z_t)) \\ &\quad + F(y_{t+1} + z_t)^T z_t + w_t^T(y - y_{t+1}) \\ &\leq F(y_{t+1} + z_t)^T(y - (y_{t+1} + z_t)) \\ &\quad + \|F(y_{t+1} + z_t)\| \|z_t\| + w_t^T(y - y_{t+1}) \\ &\leq F(y_{t+1} + z_t)^T(y - (y_{t+1} + z_t)) \\ &\quad + \sqrt{n}C\epsilon_t + w_t^T(y - y_{t+1}), \end{aligned} \quad (13)$$

where the first inequality is implied by the Cauchy-Schwartz inequality and the last inequality is obtained by boundedness of the mapping F over the set X^ϵ from Assumption 1 and by the boundedness of z_t (since $z_t = (z_{t,1}; z_{t,2}; \dots; z_{t,n})$ and $|z_{t,i}| \leq \epsilon_t$ for all i , we have $\|z_t\| \leq \sqrt{n}\epsilon_t$). From inequalities (12) and (13), we arrive at the desired relation. ■

Unlike optimization problems where the function provides a metric for distinguishing solutions, there is no immediate analog in variational inequality problems. However, one may prescribe a residual function associated with a variational inequality problem.

Definition 1 (Gap function): Let $X \subset \mathbb{R}^n$ be a nonempty and closed set. Suppose that mapping $F : X \rightarrow \mathbb{R}^n$ is defined on the set X . We define the following gap function, $G : X \rightarrow \mathbb{R} \cup \{0\}$ to measure the accuracy of a vector $x \in X$:

$$G(x) = \sup_{y \in X} F(y)^T (x - y). \quad (14)$$

We note that the gap function G is in fact also a function of the set X and the map F , but we do not use this dependency so we use G instead of $G_{X,F}$.

Lemma 4 (Properties of gap function): Consider Definition (1). We have the following properties [1]:

- (a) The gap function (14) is nonnegative for any $x \in X$.
- (b) Assume that the mapping F is bounded over the set X . Then, G is continuous at any $x \in X$.

Proposition 1 (Error bounds on the expected gap value): Consider problem (1), and let Assumptions 1 and 2 hold. Let the weighted average sequence $\{\bar{y}_k(r)\}$ be defined by

$$\bar{y}_{k+1}(r) \triangleq \frac{\sum_{t=0}^k \gamma_t^r (y_{t+1} + z_t)}{\sum_{t=0}^k \gamma_t^r}, \quad \text{for all } k \geq 0,$$

where $r \in \mathbb{R}$ is a parameter, $\{y_t\}$ is generated by algorithm (2), and the stepsize sequence $\{\gamma_t\}$ is non-increasing and $0 < \gamma_t \leq \frac{\epsilon_k}{\sqrt{5n}C}$ for all $t \geq 0$. Then, the following statements are valid:

- (a) For any $k \geq 0$, and $r \geq 1$, we have:

$$\begin{aligned} \mathbb{E}[G(\bar{y}_{k+1}(r))] &\leq \left(\sum_{t=0}^k \gamma_t^r \right)^{-1} (4M^2 \gamma_0^{r-1} \\ &+ \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + (5.5\nu^2 + 5C^2) \sum_{t=0}^k \gamma_t^{r+1}). \end{aligned} \quad (15)$$

- (b) For any $k \geq 0$, and $r < 1$, we have:

$$\begin{aligned} \mathbb{E}[G(\bar{y}_{k+1}(r))] &\leq \left(\sum_{t=0}^k \gamma_t^r \right)^{-1} \left(\frac{4M^2}{\gamma_0^{1-r}} \right. \\ &+ \frac{4M^2}{\gamma_k^{1-r}} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + (5.5\nu^2 + 5C^2) \sum_{t=0}^k \gamma_t^{r+1} \left. \right). \end{aligned} \quad (16)$$

Proof: In the first part of the proof, we allow r to be any real number and we obtain a general relation. Using the general relation, we prove parts (a) and (b) separately. Let us define u_{t+1} as

$$u_{t+1} = \Pi_X[u_t + \gamma_t w_t], \quad \text{for any } t \geq 0, \quad (17)$$

where $u_0 = x_0$. Adding and subtracting u_t , (3) yields

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 \\ &+ 2\gamma_t F(y_{t+1} + z_t)^T (y - (y_{t+1} + z_t)) + 2\sqrt{n}C\gamma_t \epsilon_t \\ &+ 2\gamma_t w_t^T (u_t - y_{t+1}) + 2\gamma_t w_t^T (y - u_t) + 5\gamma_t^2 B_t. \end{aligned} \quad (18)$$

Next, we find an upper bound for the term $2\gamma_t w_t^T (u_t - y)$:

$$\begin{aligned} \|u_{t+1} - y\|^2 &= \|\Pi_X[u_t + \gamma_t w_t] - y\|^2 \leq \|u_t + \gamma_t w_t - y\|^2 \\ &= \|u_t - y\|^2 + 2\gamma_t w_t^T (u_t - y) + \gamma_t^2 \|w_t\|^2, \end{aligned}$$

where the second relation is implied by Lemma 2(a). Thus,

$$2\gamma_t w_t^T (u_t - y) \leq \|u_t - y\|^2 - \|u_{t+1} - y\|^2 + \gamma_t^2 \|w_t\|^2.$$

The preceding relation and (18) imply that

$$\begin{aligned} \|x_{t+1} - y\|^2 &\leq \|x_t - y\|^2 \\ &+ 2\gamma_t F(y_{t+1} + z_t)^T (y - (y_{t+1} + z_t)) + 2\sqrt{n}C\gamma_t \epsilon_t \\ &+ 2\gamma_t w_t^T (u_t - y_{t+1}) + \|u_t - y\|^2 - \|u_{t+1} - y\|^2 \\ &+ 5\gamma_t^2 (B_t + 0.2\|w_t\|^2). \end{aligned} \quad (19)$$

By monotonicity of the mapping F over the set X^ϵ from Assumption 1(b) we have

$$\begin{aligned} F(y_{t+1} + z_t)^T (y - (y_{t+1} + z_t)) \\ \leq F(y)^T (y - (y_{t+1} + z_t)). \end{aligned} \quad (20)$$

From (19) and (20), and rearranging the terms we obtain

$$\begin{aligned} \gamma_t F(y)^T (y_{t+1} + z_t - y) \\ \leq 0.5\|x_t - y\|^2 - 0.5\|x_{t+1} - y\|^2 + 0.5\|u_t - y\|^2 \\ - 0.5\|u_{t+1} - y\|^2 + \sqrt{n}C\gamma_t \epsilon_t + \gamma_t w_t^T (u_t - y_{t+1}) \\ + 2.5\gamma_t^2 (B_t + 0.2\|w_t\|^2). \end{aligned}$$

Multiplying both sides of the preceding inequality by γ_t^{r-1} for some constant $r \in \mathbb{R}$ and $k \geq 0$, we have

$$\begin{aligned} \gamma_t^r F(y)^T (y_{t+1} + z_t - y) &\leq 0.5\gamma_t^{r-1}\|x_t - y\|^2 \\ &- 0.5\gamma_t^{r-1}\|x_{t+1} - y\|^2 + 0.5\gamma_t^{r-1}\|u_t - y\|^2 \\ &- 0.5\gamma_t^{r-1}\|u_{t+1} - y\|^2 + \sqrt{n}C\gamma_t^r \epsilon_t + \gamma_t^r w_t^T (u_t - y_{t+1}) \\ &+ 2.5\gamma_t^{r+1}(B_t + 0.2\|w_t\|^2). \end{aligned} \quad (21)$$

- (a) $r \geq 1$: Since $\{\gamma_t\}$ is a non-increasing sequence and $r \geq 1$, we get $\gamma_{t+1}^{r-1} \leq \gamma_t^{r-1}$. Therefore, from relation (21)

$$\begin{aligned} \gamma_t^r F(y)^T (y_{t+1} + z_t - y) &\leq 0.5\gamma_t^{r-1}\|x_t - y\|^2 - 0.5\gamma_{t+1}^{r-1}\|x_{t+1} - y\|^2 \\ &+ 0.5\gamma_t^{r-1}\|u_t - y\|^2 - 0.5\gamma_{t+1}^{r-1}\|u_{t+1} - y\|^2 + \sqrt{n}C\gamma_t^r \epsilon_t \\ &+ \gamma_t^r w_t^T (u_t - y_{t+1}) + 2.5\gamma_t^{r+1}(B_t + 0.2\|w_t\|^2). \end{aligned}$$

Summing over t from $t = 0$ to k , we obtain

$$\begin{aligned} \sum_{t=0}^k \gamma_t^r F(y)^T (y_{t+1} + z_t - y) \\ \leq \frac{\gamma_0^{r-1}}{2}\|x_0 - y\|^2 - \frac{\gamma_{k+1}^{r-1}}{2}\|x_{k+1} - y\|^2 \\ + \frac{\gamma_0^{r-1}}{2}\|u_0 - y\|^2 - \frac{\gamma_{k+1}^{r-1}}{2}\|u_{k+1} - y\|^2 + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t \\ + \sum_{t=0}^k \gamma_t^r w_t^T (u_t - y_{t+1}) + 2.5 \sum_{t=0}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

From boundedness of the set X and the triangle inequality, we have $\|u_0 - y\|^2 = \|x_0 - y\|^2 \leq 4M^2$. Thus, from the definition of $\bar{y}_{k+1}(r)$, the preceding relation implies that

$$\begin{aligned} & \left(\sum_{t=0}^k \gamma_t^r \right) F(y)^T (\bar{y}_{k+1}(r) - y) \\ & \leq 4M^2 \gamma_0^{r-1} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + \sum_{t=0}^k \gamma_t^r w_t^T (u_t - y_{t+1}) \\ & + 2.5 \sum_{t=0}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

Taking supremum over the set X with respect to y , invoking the definition of the gap function (1), we have

$$\begin{aligned} & \left(\sum_{t=0}^k \gamma_t^r \right) G(\bar{y}_{k+1}(r)) \leq 4M^2 \gamma_0^{r-1} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t \\ & + \sum_{t=0}^k \gamma_t^r w_t^T (u_t - y_{t+1}) + 2.5 \sum_{t=0}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

Taking expectations, we obtain the following:

$$\begin{aligned} & \left(\sum_{t=0}^k \gamma_t^r \right) \mathbb{E}[G(\bar{y}_{k+1}(r))] \\ & \leq 4M^2 \gamma_0^{r-1} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + \sum_{t=0}^k \gamma_t^r \mathbb{E}[w_t^T (u_t - y_{t+1})] \\ & + 2.5 \sum_{t=0}^k \gamma_t^{r+1} \mathbb{E}[B_t + 0.2\|w_t\|^2]. \end{aligned} \quad (22)$$

The algorithm (2) implies that y_{t+1} is $(\mathcal{F}_{t-1} \cup \mathcal{F}_t')$ -measurable and x_{t+1} is $(\mathcal{F}_t \cup \mathcal{F}_t')$ -measurable. Moreover, the definition of w_t in (5), and the definition of u_t in (17) imply that w_t is $(\mathcal{F}_t \cup \mathcal{F}_t')$ -measurable and u_t is $(\mathcal{F}_{t-1} \cup \mathcal{F}_t')$ -measurable. Thus, the term $u_t - y_{t+1}$ is $(\mathcal{F}_{t-1} \cup \mathcal{F}_t')$ -measurable. Also, from Assumption 2, $F(y_{t+1} + z_t) = \mathbb{E}[\Phi(y_{t+1} + z_t, \xi_t) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}]$. Therefore,

$$\begin{aligned} & \mathbb{E}[w_t^T (u_t - y_{t+1}) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}] \\ & = (u_t - y_{t+1})^T \mathbb{E}[w_t \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}] \\ & = (u_t - y_{t+1})^T \mathbb{E}[\Phi(y_{t+1} + z_t, \xi_t) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}] \\ & - (u_t - y_{t+1})^T F(y_{t+1} + z_t) = 0, \end{aligned}$$

where we use the unbiasedness of the mapping F . Taking expectation on the preceding equation, we obtain

$$\mathbb{E}[w_t^T (u_t - y_{t+1})] = 0, \quad \text{for any } t \geq 0. \quad (23)$$

Next, we estimate $\mathbb{E}[\|w_t\|^2]$. From Assumption 2, we have

$$\mathbb{E}[\|w_t\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}] \leq \nu^2, \quad \text{for any } t \geq 0.$$

Taking expectations, we obtain the following inequality:

$$\begin{aligned} & \mathbb{E}[\mathbb{E}[\|w_t\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t' \cup \{z_t\}]] \leq \nu^2 \\ & \Rightarrow \mathbb{E}[\|w_t\|^2] \leq \nu^2, \quad \text{for any } t \geq 0. \end{aligned} \quad (24)$$

In a similar fashion, we can show the following:

$$\mathbb{E}[\|w_t'\|^2] \leq \nu^2, \quad \text{for any } t \geq 0. \quad (25)$$

Next, we estimate $\mathbb{E}[\|\Delta_t\|^2]$. From the definition of Δ_t in (5):

$$\begin{aligned} & \mathbb{E}[\|\Delta_t\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \\ & = \mathbb{E}[\|F(y_{t+1} + z_t) - F_k(y_{t+1})\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \\ & = \mathbb{E}[\|F(y_{t+1} + z_t)\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \\ & + \mathbb{E}[\|F_k(y_{t+1})\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \\ & - 2\mathbb{E}[F(y_{t+1} + z_t)^T F_k(y_{t+1}) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t']. \end{aligned} \quad (26)$$

Note that from Definition 4, $F_t(y_{t+1}) = \mathbb{E}[F(y_{t+1} + z_k) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t']$. Since y_{t+1} is $(\mathcal{F}_{t-1} \cup \mathcal{F}_t')$ -measurable, we observe that $F_t(y_{t+1})$ is also $(\mathcal{F}_{t-1} \cup \mathcal{F}_t')$ -measurable. Therefore, from relation (26) we have

$$\begin{aligned} & \mathbb{E}[\|\Delta_t\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \\ & = \mathbb{E}[\|F(y_{t+1} + z_t)\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] + \|F_t(y_{t+1})\|^2 \\ & - 2\mathbb{E}[F(y_{t+1} + z_t) \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t']^T F_k(y_{t+1}) \\ & = \mathbb{E}[\|F(y_{t+1} + z_t)\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] + \|F_t(y_{t+1})\|^2 \\ & - 2F_t(y_{t+1})^T F_t(y_{t+1}) \\ & \leq \mathbb{E}[\|F(y_{t+1} + z_t)\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \leq C^2, \end{aligned}$$

where the last inequality is obtained using boundedness of the mapping F over the set X^ϵ . Taking expectations over the preceding relation, we get

$$\begin{aligned} & \mathbb{E}[\mathbb{E}[\|\Delta_t\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t']] \leq C^2 \\ & \Rightarrow \mathbb{E}[\|\Delta_t\|^2] \leq C^2, \quad \text{for any } t \geq 0. \end{aligned} \quad (27)$$

In a similar fashion, we can show that

$$\mathbb{E}[\|\Delta_t'\|^2] \leq C^2, \quad \text{for any } t \geq 0. \quad (28)$$

In conclusion, invoking relations (23), (24), (25), (27), and (28), from relation (22) we conclude with the following:

$$\begin{aligned} & \left(\sum_{t=0}^k \gamma_t^r \right) \mathbb{E}[G(\bar{y}_{k+1}(r))] \\ & \leq 4M^2 \gamma_0^{r-1} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + (5.5\nu^2 + 5C^2) \sum_{t=0}^k \gamma_t^{r+1} \end{aligned}$$

implying the desired result (15).

(b) $r < 1$: Consider relation (20). Adding and subtracting the terms $0.5\gamma_{t-1}^{r-1}\|x_t - y\|^2$ and $0.5\gamma_{t-1}^{r-1}\|u_t - y\|^2$, we can write

$$\begin{aligned} & \gamma_t^r F(y)^T (y_{t+1} + z_t - y) \\ & \leq \frac{\|x_t - y\|^2}{2\gamma_{t-1}^{1-r}} - \frac{\|x_{t+1} - y\|^2}{2\gamma_t^{1-r}} + \frac{\|u_t - y\|^2}{2\gamma_{t-1}^{1-r}} - \frac{\|u_{t+1} - y\|^2}{2\gamma_t^{1-r}} \\ & + \left(\frac{1}{2\gamma_t^{1-r}} - \frac{1}{2\gamma_{t-1}^{1-r}} \right) (\|x_t - y\|^2 + \|u_t - y\|^2) \\ & + \sqrt{n}C \gamma_t^r \epsilon_t + \gamma_t^r w_t^T (u_t - y_{t+1}) \\ & + 2.5\gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

Boundedness of X implies that $\|x_t - y\|^2 + \|u_t - y\|^2 \leq 8M^2$.

By summing over t from $t = 1$ to k , we obtain

$$\begin{aligned} & \sum_{t=1}^k \gamma_t^r F(y)^T (y_{t+1} + z_t - y) \\ & \leq \frac{\|x_1 - y\|^2}{2\gamma_0^{1-r}} - \frac{\|x_{k+1} - y\|^2}{2\gamma_k^{1-r}} + \frac{\|u_1 - y\|^2}{2\gamma_0^{1-r}} - \frac{\|u_{k+1} - y\|^2}{2\gamma_k^{1-r}} \\ & + 4M^2 \left(\frac{1}{\gamma_k^{1-r}} - \frac{1}{\gamma_0^{1-r}} \right) + \sqrt{n}C \sum_{t=1}^k \gamma_t^r \epsilon_t \\ & + \sum_{t=1}^k \gamma_t^r w_t^T (u_t - y_{t+1}) + 2.5 \sum_{t=1}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

By letting $t = 0$ in relation (20), and then adding the resulting inequality to the preceding inequality, we obtain the following:

$$\begin{aligned} & \sum_{t=0}^k \gamma_t^r F(y)^T (y_{t+1} + z_t - y) \\ & \leq \frac{\|x_0 - y\|^2}{2\gamma_0^{1-r}} - \frac{\|x_{k+1} - y\|^2}{2\gamma_k^{1-r}} + \frac{\|u_0 - y\|^2}{2\gamma_0^{1-r}} - \frac{\|u_{k+1} - y\|^2}{2\gamma_k^{1-r}} \\ & + 4M^2 \left(\frac{1}{\gamma_k^{1-r}} - \frac{1}{\gamma_0^{1-r}} \right) + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t \\ & + \sum_{t=0}^k \gamma_t^r w_t^T (u_t - y_{t+1}) + 2.5 \sum_{t=0}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

Invoking boundedness of the set X again, we obtain

$$\begin{aligned} & \sum_{t=0}^k \gamma_t^r F(y)^T (y_{t+1} + z_t - y) \\ & \leq \frac{4M^2}{\gamma_0^{1-r}} + \frac{4M^2}{\gamma_k^{1-r}} + \sqrt{n}C \sum_{t=0}^k \gamma_t^r \epsilon_t + \sum_{t=0}^k \gamma_t^r w_t^T (u_t - y_{t+1}) \\ & + 2.5 \sum_{t=0}^k \gamma_t^{r+1} (B_t + 0.2\|w_t\|^2). \end{aligned}$$

The remainder of the proof can be done in a similar fashion to the proof of part (a). \blacksquare

Theorem 1 (Optimal rate of convergence for $\bar{y}_k(r)$):

Under assumptions of Proposition 1, consider the weighted average sequence $\{\bar{y}_k(r)\}$ of the sequence $\{y_t\}$ generated by algorithm (2), where

$$\gamma_t = \frac{\gamma_0}{\sqrt{t+1}}, \quad \epsilon_t = \frac{\epsilon_0}{\sqrt{t+1}}, \quad \text{with } \gamma_0 \leq \frac{\epsilon_0}{\sqrt{5n}C}.$$

Then, when $r < 1$, we have

$$\mathbb{E}[G(\bar{y}_k(r))] \leq \frac{\theta_r}{\sqrt{k}}, \quad \text{for all } k \geq 1,$$

where $\theta_r \triangleq 4(2-r)\gamma_0^{-r}M^2 + \frac{2-r}{1-r}(\sqrt{n}C\epsilon_0 + \gamma_0(5.5\nu^2 + 5C^2))$.

Proof: Let us define

$$\text{Term A} \triangleq \frac{(k+1)^{\frac{1-r}{2}}}{\sum_{t=0}^k (t+1)^{-\frac{r}{2}}}, \quad \text{Term B} \triangleq \frac{\sum_{t=0}^k (t+1)^{-\frac{(1+r)}{2}}}{\sum_{t=0}^k (t+1)^{-\frac{r}{2}}}.$$

Consider Proposition 1(b). Note that $\gamma_k \leq \gamma_0$ and $r < 1$ imply that $\frac{4M^2}{\gamma_0^{1-r}} \leq \frac{4M^2}{\gamma_k^{1-r}}$. From the definitions of Term A and B, the relation Proposition 1(b) implies that

$$\mathbb{E}[G(\bar{y}_{k+1}(r))] \leq R_1(\text{Term A}) + R_2(\text{Term B}), \quad (29)$$

where $R_1 \triangleq 8\gamma_0^{-r}M^2$ and $R_2 \triangleq \sqrt{n}C\epsilon_0 + \gamma_0(5.5\nu^2 + 5C^2)$. We make use of the following relation in our analysis:

$$\int_0^{k+1} (x+1)^{-p} dx \leq \sum_{t=0}^k (t+1)^{-p} \leq 1 + \int_0^k (x+1)^{-p} dx,$$

where $p \in \mathbb{R}$ and $k \geq 0$. From this relation, it follows that

$$\begin{aligned} \text{Term A} & \leq \frac{(k+1)^{\frac{1-r}{2}}}{\int_0^{k+1} (x+1)^{-\frac{r}{2}} dx} = \left(1 - \frac{r}{2}\right) \frac{(k+1)^{\frac{1-r}{2}}}{(k+1)^{(1-\frac{r}{2})}} \\ & = \frac{2-r}{2\sqrt{k+1}}, \end{aligned}$$

$$\begin{aligned} \text{Term B} & \leq \frac{\int_0^k (x+1)^{-\frac{(1+r)}{2}} dx}{\int_0^{k+1} (x+1)^{-\frac{r}{2}} dx} = \left(\frac{2-r}{1-r}\right) \frac{(k+1)^{\frac{1-r}{2}}}{(k+1)^{(1-\frac{r}{2})}} \\ & = \frac{2-r}{(1-r)\sqrt{k+1}}. \end{aligned}$$

In conclusion, replacing the preceding bounds for Term A and Term B in relation (29), we get the desired result. \blacksquare

III. RECURSIVE STEPSIZE AND SMOOTHING

Motivated by the little guidance on the choice of a diminishing stepsize, in this section, we consider algorithm (2) and assume that the stepsize γ_t and the smoothing sequence ϵ_t are given by

$$\begin{cases} \gamma_0^* = \frac{2\alpha M}{q} \\ \gamma_{t+1}^* = \gamma_t^* \left(1 - \frac{\alpha}{2M}\gamma_t^*\right) \end{cases}, \quad \begin{cases} \epsilon_0^* = \frac{2\alpha\beta M}{q} \\ \epsilon_{t+1}^* = \epsilon_t^* \left(1 - \frac{\alpha}{2\beta M}\epsilon_t^*\right) \end{cases}, \quad (30)$$

where β is a constant such that $\beta > \max\left\{\sqrt{5n}C, \frac{\alpha^2 - 5(C^2 + \nu^2)}{2\sqrt{n}C}\right\}$ and $q \triangleq \alpha + 2C^2\alpha + 2\alpha\nu^2 + 2(\alpha + 1)\beta\sqrt{n}C + 5(C^2 + \nu^2)$. Our goal is to analyze the convergence of $\{x_k\}$ to the solution of problem (1).

Definition 2 (Weak-sharpness property): Consider $\text{VI}(X, F)$ where $X \subset \mathbb{R}^n$ and $F : X \rightarrow \mathbb{R}^n$ is a continuous mapping. Let X^* denote the solution set of $\text{VI}(X, F)$. The problem has a weak-sharpness property with parameter $\alpha > 0$, if for all $x \in X$ and all $x^* \in X^*$

$$F(x^*)^T (x - x^*) \geq \alpha \text{dist}(x, X^*). \quad (31)$$

Lemma 5 (A recursive error bound): Consider algorithm (2). Let Assumption 1 and 2 hold and suppose mapping F is strictly monotone over X and $\gamma_k \leq \frac{\epsilon_k}{\sqrt{5n}C}$ for any $k \geq 0$. Moreover, assume that problem (1) has the weak-sharpness property with parameter $\alpha > 0$. Then, problem (1) has a unique solution, x^* , and the following relation holds:

$$\begin{aligned} & \mathbb{E}[\|x_{t+1} - x^*\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] \\ & \leq \left(1 - \frac{\alpha\gamma_t}{M}\right) \|x_t - x^*\|^2 + q_1 \epsilon_t \gamma_t + q_2 \gamma_t^2, \end{aligned} \quad (32)$$

where $q_1 \triangleq \alpha(1 + 2(C^2 + \nu^2)) + 5(C^2 + \nu^2)$ and $q_2 \triangleq 2(1 + \alpha)\beta\sqrt{n}C$.

Proof: Since the mapping F is strictly monotone over the closed and convex set X , the problem $\text{VI}(X, F)$ has at most one solution. From non-emptiness of the solution set of the problem (1), we conclude that it has a unique solution. Let

x^* be such a solution. From relation (6), for $y = x^*$ and the monotonicity property of the mapping F we have

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &\leq \|x_t - x^*\|^2 + 2\gamma_t F(x^*)^T (x^* - (y_{t+1} + z_t)) \\ &\quad + 2\sqrt{n}C\gamma_t\epsilon_t + 2\gamma_t w_t^T (x^* - y_{t+1}) + 5\gamma_t^2 B_t. \end{aligned}$$

Invoking the weak-sharpness property and the uniqueness of the solution set, we obtain

$$\begin{aligned} \|x_{t+1} - x^*\|^2 &\leq \|x_t - x^*\|^2 - 2\alpha\gamma_t \|y_{t+1} + z_t - x^*\| \\ &\quad + 2\sqrt{n}C\gamma_t\epsilon_t + 2\gamma_t w_t^T (x^* - y_{t+1}) + 5\gamma_t^2 B_t. \end{aligned}$$

Taking conditional expectation, we obtain

$$\begin{aligned} &\mathbb{E}[\|x_{t+1} - x^*\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] \\ &\leq \|x_t - x^*\|^2 - 2\alpha\gamma_t \mathbb{E}[\|y_{t+1} + z_t - x^*\| \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] \\ &\quad + 2\sqrt{n}C\gamma_t\epsilon_t + 5\gamma_t^2 (C^2 + \nu^2), \end{aligned} \quad (33)$$

where we used $\mathbb{E}[w_t^T (x^* - y_{t+1}) \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] = 0$. Using the triangle inequality, we can write

$$\begin{aligned} \|y_{t+1} + z_t - x^*\| &\geq \|y_{t+1} - x^*\| - \|z_t\| \\ &\Rightarrow -2\alpha\gamma_t \|y_{t+1} + z_t - x^*\| \leq -2\alpha\gamma_t \|y_{t+1} - x^*\| \\ &\quad + 2\alpha\gamma_t \|z_t\| \leq -2\alpha\gamma_t \|y_{t+1} - x^*\| + 2\alpha\sqrt{n}C\gamma_t\epsilon_t. \end{aligned} \quad (34)$$

Next, we find a lower bound for the term $\|y_{t+1} - x^*\|$. Using the triangle inequality we have

$$\begin{aligned} \|y_{t+1} - x^*\| &\geq \|x_t - x^*\| - \|y_{t+1} - x_t\| \\ &\geq \frac{\|x_t - x^*\|^2}{2M} - \|y_{t+1} - x_t\|, \end{aligned} \quad (35)$$

where in the last inequality we used the boundedness of the set X . We also have

$$\begin{aligned} \|y_{t+1} - x_t\| &= \|\Pi_X [x_t - \gamma_t \Phi(x_t + z'_t, \xi'_t)] - x_t\| \\ &\leq \|x_t - \gamma_t \Phi(x_t + z'_t, \xi'_t) - x_t\| = \gamma_t \|\Phi(x_t + z'_t, \xi'_t)\| \\ &= \gamma_t \|F(x_t + z'_t) + w'_t\| \leq 0.5\gamma_k (1^2 + \|F(x_t + z'_t) + w'_t\|^2) \\ &\leq 0.5\gamma_k (1^2 + 2\|F(x_t + z'_t)\|^2 + 2\|w'_t\|^2) \\ &\leq 0.5\gamma_k (1 + 2C^2 + 2\|w'_t\|^2). \end{aligned} \quad (36)$$

Relations (34), (35), and (36) imply that

$$\begin{aligned} -2\alpha\gamma_t \|y_{t+1} + z_t - x^*\| &\leq -\frac{\alpha\gamma_t}{M} \|x_t - x^*\|^2 + \alpha\gamma_t^2 \\ &\quad + 2C^2\alpha\gamma_t^2 + 2\alpha\gamma_t^2 \|w'_t\|^2 + 2\alpha\sqrt{n}C\gamma_t\epsilon_t. \end{aligned}$$

From the preceding relation, and (33) we have

$$\begin{aligned} &\mathbb{E}[\|x_{t+1} - x^*\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] \\ &\leq \left(1 - \frac{\alpha\gamma_t}{M}\right) \|x_t - x^*\|^2 + \alpha\gamma_t^2 + 2C^2\alpha\gamma_t^2 + 2\alpha\gamma_t^2 \nu^2 \\ &\quad + 2\alpha\sqrt{n}C\gamma_t\epsilon_t + 2\sqrt{n}C\gamma_t\epsilon_t + 5\gamma_t^2 (C^2 + \nu^2). \end{aligned}$$

Replacing ϵ_t by $\beta\gamma_t$, we obtain the desired relation. ■

We use the following Lemma in establishing the almost-sure convergence (cf. Lemma 10, page 49 [15]).

Lemma 6: Let $\{v_k\}$ be a sequence of nonnegative random variables, where $\mathbb{E}[v_0] < \infty$, and let $\{\alpha_k\}$ and $\{\mu_k\}$ be

deterministic scalar sequences such that:

$$\begin{aligned} \mathbb{E}[v_{k+1} \mid v_0, \dots, v_k] &\leq (1 - \alpha_k)v_k + \mu_k \quad \text{a.s. for all } k \geq 0, \\ 0 &\leq \alpha_k \leq 1, \quad \mu_k \geq 0, \\ \sum_{k=0}^{\infty} \alpha_k &= \infty, \quad \sum_{k=0}^{\infty} \mu_k < \infty, \quad \lim_{k \rightarrow \infty} \frac{\mu_k}{\alpha_k} = 0. \end{aligned}$$

Then, $v_k \rightarrow 0$ almost surely.

Theorem 2 (Optimal rate of convergence for x_k):

Consider algorithm (2). Let Assumption 1 and 2 hold, let mapping F be strictly monotone over X , and assume that problem (1) has the weak-sharpness property with parameter $\alpha > 0$. Suppose the sequences γ_t and ϵ_t are given by the recursive relations (30). Then, problem (1) has a unique solution, x^* , and the following results hold:

- (a) The sequence $\{x_t\}$ generated by the algorithm (2) converges to the solution of problem (1) almost surely as $k \rightarrow \infty$.
- (b) The sequence $\{x_t\}$ generated by the algorithm (2) converges to the solution of problem (1) in a mean-squared sense. More precisely, we have

$$\mathbb{E}[\|x_t - x^*\|^2] \leq \left(\frac{4qM^2}{\alpha^2}\right) \frac{1}{t}, \quad \text{for all } t \geq 1.$$

Proof: (a) First we show that $\epsilon_t^* = \beta\gamma_t^*$ for any $t \geq 0$. From (30), we have $\epsilon_0^* = \frac{2\alpha\beta M}{q} = \beta\gamma_0^*$, implying that the relation holds for $t = 0$. Assume that the relation holds for some fixed t . We show that it holds for $t + 1$. We have

$$\begin{aligned} \epsilon_{t+1}^* &= \epsilon_t^* \left(1 - \frac{\alpha}{2\beta M} \epsilon_t^*\right) = \beta\gamma_t^* \left(1 - \frac{\alpha}{2\beta M} \beta\gamma_t^*\right) \\ &= \beta\gamma_{t+1}^* \end{aligned}$$

Therefore, we conclude that $\epsilon_t^* = \beta\gamma_t^*$ for any $t \geq 0$. Since we assumed $\beta > \sqrt{5n}C$, we get $\epsilon_t^* \geq \sqrt{5n}C\gamma_t^*$ or equivalently, $\gamma_t < \frac{\epsilon_t^*}{\sqrt{5n}C}$ for any $t \geq 0$. Next, we show that $\{\gamma_t^*\}$ is a decreasing sequence with strictly positive elements. We have

$$\begin{aligned} \frac{\alpha^2 - 5(C^2 + \nu^2)}{2\sqrt{n}C} &< \beta \Rightarrow \alpha^2 < 5(C^2 + \nu^2) + 2\sqrt{n}C\beta \\ &\Rightarrow \alpha^2 < \alpha(1 + 2C^2 + 2\nu^2 + 2\sqrt{n}C\beta) + 5(C^2 + \nu^2) \\ &\quad + 2\sqrt{n}C\beta = q \Rightarrow \frac{\alpha}{q} < \frac{1}{\alpha} \Rightarrow \gamma_0^* < \frac{2M}{\alpha}. \end{aligned}$$

Using the preceding relation we obtain $\gamma_1^* = \gamma_0^*(1 - \frac{\alpha}{2M}\gamma_0^*) < \gamma_0^* < \frac{2M}{\alpha}$ and $\gamma_1^* > 0$. Following the same approach, induction implies that

$$0 < \dots < \gamma_2^* < \gamma_1^* < \gamma_0^* < \frac{2M}{\alpha}, \quad (37)$$

verifying that $\{\gamma_t^*\}$ is a decreasing sequence with strictly positive terms. Therefore, all the conditions of Lemma 5 hold showing that for the unique solution x^* , we have

$$\begin{aligned} &\mathbb{E}[\|x_{t+1} - x^*\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}'_t] \\ &\leq \left(1 - \frac{\alpha\gamma_t}{M}\right) \|x_t - x^*\|^2 + q_1\epsilon_t\gamma_t + q_2\gamma_t^2 \\ &\leq \left(1 - \frac{\alpha\gamma_t}{M}\right) \|x_t - x^*\|^2 + \beta q_1\gamma_t^2 + q_2\gamma_t^2, \end{aligned}$$

where we used the relation $\epsilon_t^* = \beta\gamma_t^*$. From definition of q_1 , q_2 and q , we have $q = \beta q_1 + q_2$. Thus, from the preceding relation we obtain

$$\mathbb{E}[\|x_{t+1} - x^*\|^2 \mid \mathcal{F}_{t-1} \cup \mathcal{F}_t'] \leq \left(1 - \frac{\alpha\gamma_t^*}{M}\right) \|x_t - x^*\|^2 + q\gamma_t^2, \quad \text{for all } t \geq 0.$$

The next step is to show $\sum_{t=0}^{\infty} \gamma_t^* = \infty$ and $\sum_{t=0}^{\infty} (\gamma_t^*)^2 < \infty$. The proof of these relations can be found in our prior work (cf. Prop. 3 in [12]). The last step of the proof is applying Lemma 6 on the preceding inequality. We verified that all the conditions of Lemma 6 are satisfied for $v_t \triangleq \|x_t - x^*\|^2$, $\alpha_t \triangleq \frac{\alpha}{2M}\gamma_t^*$, $\mu_t \triangleq q(\gamma_t^*)^2$. Therefore, x_t converges to x^* almost surely.

(b) In the first part of the proof, using induction on t , we show that using the sequences γ_t^* and ϵ_t^* , we have

$$\mathbb{E}[\|x_t - x^*\|^2] \leq \frac{2Mq}{\alpha}\gamma_t^*, \quad \text{for all } t \geq 0. \quad (38)$$

For $t = 0$, the relation becomes $\mathbb{E}[\|x_0 - x^*\|^2] \leq \frac{2Mq}{\alpha}\gamma_0^* = 4M^2$. This holds because $\mathbb{E}[\|x_0 - x^*\|^2] \leq \mathbb{E}[2\|x_0\|^2 + 2\|x^*\|^2] \leq 4M^2$. Let us assume that relation (38) holds for t . Then, taking expectation from (32) and the definition of q we can write

$$\mathbb{E}[\|x_{t+1} - x^*\|^2] \leq \left(1 - \frac{\alpha\gamma_t^*}{M}\right) \mathbb{E}[\|x_t - x^*\|^2] + q(\gamma_t^*)^2, \quad \text{for all } t \geq 0.$$

Using the induction hypothesis, from the preceding relation it follows

$$\begin{aligned} \mathbb{E}[\|x_{t+1} - x^*\|^2] &\leq \left(1 - \frac{\alpha\gamma_t^*}{M}\right) \frac{2Mq}{\alpha}\gamma_t^* + q(\gamma_t^*)^2 \\ &= \frac{2Mq}{\alpha}\gamma_t^* \left(1 - \frac{\alpha\gamma_t^*}{M} + \frac{\alpha\gamma_t^*}{2M}\right) = \frac{2Mq}{\alpha}\gamma_{t+1}^*. \end{aligned}$$

Therefore, the relation (38) holds for $t + 1$, implying that it holds for any $t \geq 0$. In the second part of the proof, we show that

$$\gamma_t^* \leq \frac{2M}{\alpha t}, \quad \text{for all } t \geq 0. \quad (39)$$

From definition of the sequence $\{\gamma_t^*\}$ we have for $t \geq 0$,

$$\frac{1}{\gamma_{t+1}^*} = \frac{1}{\gamma_t^* \left(1 - \frac{\alpha\gamma_t^*}{2M}\right)} = \frac{1}{\gamma_t^*} + \frac{\frac{\alpha}{2M}}{1 - \frac{\alpha\gamma_t^*}{2M}}.$$

Summing up from $t = 0$ to k we obtain

$$\frac{1}{\gamma_{k+1}^*} = \frac{1}{\gamma_0^*} + \frac{\alpha}{2M} \sum_{t=0}^k \frac{1}{1 - \frac{\alpha\gamma_t^*}{2M}} > \frac{\alpha}{2M} \sum_{t=0}^k \frac{1}{1 - \frac{\alpha\gamma_t^*}{2M}}. \quad (40)$$

From the Cauchy-Schwarz inequality, $\frac{1}{\frac{1}{n} \sum_{k=1}^n a_k} \leq \frac{1}{n} \sum_{k=1}^n \frac{1}{a_k}$ holds for arbitrary positive numbers a_1, a_2, \dots, a_n . Thus, for the terms $1 - \frac{\alpha\gamma_t^*}{2M}$ we get

$$\begin{aligned} \left(\frac{1}{k+1} \sum_{t=0}^k \frac{1}{1 - \frac{\alpha\gamma_t^*}{2M}} \right)^{-1} &\leq \frac{1}{k+1} \sum_{t=0}^k \left(1 - \frac{\alpha\gamma_t^*}{2M}\right) \\ &< \frac{\sum_{t=0}^k 1}{k+1} = 1 \Rightarrow \sum_{t=0}^k \frac{1}{1 - \frac{\alpha\gamma_t^*}{2M}} > k+1. \end{aligned}$$

The preceding relation and (40) imply that the inequality (39) holds. In conclusion, using the two relations (38) and (39), we obtain the desired result. ■

IV. CONCLUDING REMARKS

We presented two robust variants of a stochastic extragradient method for solving monotone stochastic VIs by utilizing a smoothing technique. First, using a new class of choices for the weights in an averaging scheme, we show that a suitably defined gap function converges to zero at rate of $\mathcal{O}\left(\frac{1}{\sqrt{k}}\right)$. Second, we develop a recursive rule for updating stepsize and smoothing parameters and we show both the almost-sure convergence and that the rate in mean-squared sense is optimal. Importantly, this scheme allows for tuning the steplength sequence to problem parameters.

REFERENCES

- [1] F. Facchinei and J.-S. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems. Vols. I, II*, ser. Springer Series in Operations Research. New York: Springer-Verlag, 2003.
- [2] R. Rockafellar and R.-B. Wets, *Variational Analysis*. Berlin: Springer, 1998.
- [3] A. Nemirovski, A. Juditsky, G. Lan, and A. Shapiro, "Robust stochastic approximation approach to stochastic programming," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1574–1609, 2009.
- [4] F. Yousefian, A. Nedić, and U. V. Shanbhag, "A regularized smoothed stochastic approximation (rssa) algorithm for stochastic variational inequality problems," *Proceedings of the 2013 Winter Simulation Conference*, pp. 933–944, 2013.
- [5] A. Juditsky, A. Nemirovski, and C. Tauvel, "Solving variational inequalities with stochastic mirror-prox algorithm," *Stochastic Systems*, vol. 1, no. 1, pp. 17–58, 2011.
- [6] C. D. Dang and G. Lan, "On the convergence properties of non-Euclidean extragradient methods for variational inequalities with generalized monotone operators," *Technical Report, Department of Industrial and Systems Engineering, University of Florida*, 2013.
- [7] A. Juditsky, A. Nemirovski, and C. Tauvel, "Prox-method with rate of convergence $\mathcal{O}(1/t)$ for variational inequalities with lipschitz continuous monotone operators and smooth convex-concave saddle point problems," *SIAM Journal on Optimization*, vol. 15, pp. 229–251, 2004.
- [8] A. Kannan and U. V. Shanbhag, "The pseudomonotone stochastic variational inequality problem: Analytical statements and stochastic extragradient schemes," *Proceedings of the American Control Conference (ACC)*, 2014.
- [9] A. Nedić and S. Lee, "On stochastic subgradient mirror-descent algorithm with weighted averaging," *SIAM Journal on Optimization*, vol. 24, no. 1, pp. 84–107, 2014.
- [10] S. Ghadimi and G. Lan, "Optimal stochastic approximation algorithms for strongly convex stochastic composite optimization, part I: a generic algorithmic framework," *SIAM Journal on Optimization*, vol. 22, no. 4, pp. 1469–1492, 2012.
- [11] V. A. Steklov, "Sur les expressions asymptotiques decertaines fonctions dfinies par les quations diffrentielles du second ordre et leurs applications au probleme du developement d'une fonction arbitraire en sries procndant suivant les diverses fonctions," *Comm. Charkov Math. Soc.*, vol. 2, no. 10, pp. 97–199, 1907.
- [12] F. Yousefian, A. Nedić, and U. V. Shanbhag, "On stochastic gradient and subgradient methods with adaptive steplength sequences," *Automatica*, vol. 48, no. 1, pp. 56–67, 2012, an extended version of the paper available at: <http://arxiv.org/abs/1105.4549>.
- [13] —, "Distributed adaptive steplength stochastic approximation schemes for cartesian stochastic variational inequality problems," 2014, submitted, <http://arxiv.org/abs/1301.1711>.
- [14] D. P. Bertsekas, A. Nedić, and A. E. Ozdaglar, *Convex analysis and optimization*. Belmont, MA: Athena Scientific, 2003.
- [15] B. Polyak, *Introduction to Optimization*. New York: Optimization Software, Inc., 1987.